On Closed Fully Stable Acts

Shaymaa Amer Abdul-Kareem, Muntaha Abdul-Razaq Hasan

Department of Mathematics, College of Basic Education, Mustansiriya University, Baghdad, Iraq

Abstract:
The purpose of this paper is to introduce and study closed fully stable acts as a concept of generalization of fully stable acts. Some properties and characterizations of class of closed fully stable acts are considered. The relations between this class and other well classes of acts, like quasi-injective acts and other classes of injectivity are discussed.

Keywords: closed fully stable act, extending act, quasi-injective act, closed fully pseudo act.

2010 Mathematics Subject Classification: 20M30.

1-Introduction:
Throughout this work, S is a monoid with zero element and every S-act is unitary right S-act with zero element Θ which denoted by M_s. For more details about S-acts we refer the reader to the reference [1].

M.S. Abbas introduced in [2] a class of modules which was called a fully stable module which prompted Hiba to give the corresponding definition for S-acts as follows: let M_s be an S-act. A subact N of M_s is called stable, if f(N) ⊆ N for each S-homomorphism f : N → M_s. An S-act M is called fully stable in case each subact of M_s is stable. A monoid S is fully stable if it is a fully stable S-act [3].

A subsystem N of S-system M_s is called closed if it has no proper ∩-large in M_s that is the only solution of N ↘ [1] L ↗ x M_s is N = L [4].

The concept of fully stable S-act motivated us to introduce and study a generalization of its which is closed fully stable concept as a class of acts, and give several characterizations of these acts. Apart of this paper devoted to study the relations between this class and some acts like quasi-injective , Baer's criterion and extending acts.

2- Closed fully stable acts:
Definition(2.1): Let M_s be a right S-act. A closed subact N of M_s is called closed stable if f(N) ⊆ N for each S-homomorphism f : N → M_s. An S-act M_s is called closed fully stable act (for short cl-fully stable) in case each closed subact of M_s is stable. A monoid S is closed fully stable if it is cl-fully stable S-act.

Remarks and Examples(2.2):
1- Every fully stable act is cl-fully stable act, but the converse is not true in general for example Z with multiplication as Z-act is cl-fully stable act but not fully stable.

2- Isomorphism act to cl-fully stable act is cl-fully stable act.
3. Every cl-fully stable is fully invariant, but the converse is not true in general for example Z with multiplication as Z-act is cl-fully stable act but not fully invariant, for this if we define \( f: 2Z \to Z \) by \( f(2n)=3n \), then it is clear that \( f(2Z) \not\subseteq 2Z \) since \( f(2.1)=3 \not\in f(2Z) \).

The following lemmas explain under which conditions the subact inherit the property of closed fully stable act:

**Lemma (2.4):** Every closed subact of closed fully stable act is closed fully stable.

**Proof:** Let \( M_s \) be closed fully stable S-act and \( N \) be closed subact of \( M_s \). Let \( H \) be closed subact of \( N \). Then \( H \) is closed subact of \( M_s \) by lemma (2.4) in [5]. Let \( f:H\to N \) be an S-homomorphism and \( i_N:N\to M_s \) be the inclusion map, so \( i_N \circ f:H\to M_s \) be an S-homomorphism. As \( M_s \) is cl-fully stable act, so \( i_N \circ f(H) \subseteq H \) and this implies that \( f(H) \subseteq H \). Thus \( N \) is cl-fully stable act. ■

**Lemma (2.5):** Every retract subact of closed fully stable act is closed fully stable.

**Proof:** By remarks and examples (2.2)(4) in [4] and by lemma(2.4). ■

**Proposition(2.6):** Let \( M_s \) be an S-act in which every closed is a retract of \( M_s \). If End(\( M_s \)) is commutative, then \( M_s \) is cl-fully stable act.

**Proof:** Let \( N \) be any closed subact of \( M_s \) and \( f:N\to M_s \) be an S-homomorphism. Then, by assumption there exists a subact \( H \) such that \( M_s=H\cup N \). \( f \) can be extended to an S-homomorphism \( g:M_s\to M_s \) by putting \( g(h)=\theta \) for each \( h \in H \). Define \( K:M_s\to M_s \) by \( K(x,y)=x \) for each \( x \in N \) and \( y \in H \). Let \( f(x)=(y,1) \) for some \( y \in N \) and \( 1 \in H \). Now \( K \circ g(w)=K(g(x,h))=K(g(x))=K(f(x))=K(y,1)=y \). On the other hand \( g \circ K(w)=g(K(x,y))=g(x)=f(x)=(y,1) \). Since \( K \circ g=g \circ K \), then \( (y,1)=(y,0) \) and \( 1=0 \) which is a contradiction. Thus \( f(x) \in N \) and therefore \( f(N) \subseteq N \), hence \( M_s \) is cl-fully stable. ■

Because in extending acts every closed subact is a retract in [5], then we have:

**Corollary (2.7):** Let \( M_s \) be extending act. Then \( M_s \) is cl-fully stable if and only if End(\( M_s \)) is commutative. ■

**Proposition(2.8):** Let \( M_s \) be an S-act such that every closed subact is a retract of \( M_s \). If End(\( M_s \)) is cl-fully stable monoid, then \( M_s \) is cl-fully stable act.

**Proof:** Let \( N \) be closed subact of \( M_s \alpha:N\to M_s \). Consider \( K=\text{Hom}(M_s,N) \), is closed right ideal of End(\( M_s \)). Define \( \beta:K\to\text{End}(M_s) \) by \( \beta(f)=\alpha \circ f \) for each \( f \in K \). Clearly, \( \beta(f) \in \text{End}(M_s) \), moreover \( \beta \) is End(\( M_s \))-homomorphism. Since End(\( M_s \)) is cl-fully stable, so \( \beta(K) \subseteq K \). That is for each \( f \in K \), \( \alpha \circ f \in K \) and then \( \alpha \circ f:M_s\to N \). But \( N \) is a retract of \( M_s \), then the natural projection \( \pi_N \) of \( M_s \) onto \( N \) is in \( K \), hence \( \alpha \circ \pi_N \in K \). That is \( \alpha \circ \pi_N :M_s\to N \), since \( \pi_N \) is onto, so \( \alpha:N\to N \) or \( \alpha(N)\subseteq N \). Thus \( M_s \) is cl-fully stable act. ■

**Corollary(2.9):** Let \( M_s \) be extending S-act. If End(\( M_s \)) is cl-fully stable monoid, then \( M_s \) is cl-fully stable act. ■

**Proposition(2.10):** Let \( M_s \) be S-act such that every subact is closed. Then \( M_s \) is cl-fully stable act if and only if End(\( M_s \)) is cl-fully stable monoid.

**Proof:** Let \( M_s \) be cl-fully stable and extending S-act. Let \( I=\text{Hom}(M_s,N) \) be closed right ideal of End(\( M_s \)) and \( \alpha:I\to\text{End}(M_s) \). As \( M_s \) is cl-fully stable, so for each S-
homomorphism $f : N \rightarrow M_s$, $f(N) \subseteq N$. Then, it is clear that for each $g \in I$, $f \circ g \in \text{End}(M_s)$. Since $\text{End}(M_s)$ is commutative by corollary(2.7), so $f \circ g = g \circ f$. This means that $f$ and $g$ are isomorphisms. Then, since $f : N \rightarrow N$, we have $f \circ g \in I$. This implies that $f \circ g \in \text{End}(M_s) = \alpha(I)$ and on the other hand $f \circ g \in I$. Therefore, $\alpha(I) \subseteq I^s$.

**Corollary (2.11):** Let $M_s$ be quasi injective $S$-act. Then $M_s$ is $cl$-fully stable act if and only if $\text{End}(M_s)$ is $cl$-fully stable monoid.

**Corollary (2.12):** Let $M_s$ be quasi injective $S$-act with $\psi_M = I$. Then the following statements are equivalent:

1. $M_s$ is $cl$-fully stable act;
2. $\text{End}(M_s)$ is commutative monoid;
3. $\text{End}(M_s)$ is $cl$-fully stable act.

**Proof:** (1$\rightarrow$2) As quasi injective $S$-act with $\psi_M = i$ is extending act by proposition (4.1) in [5], so by corollary (2.7) $\text{End}(M_s)$ is commutative monoid.

(2$\rightarrow$3) As previous argument in (1$\rightarrow$2), we obtain $M_s$ is extending, so by corollary(2.7), $M_s$ is $cl$-fully stable act and then by corollary(2.11) implies that $\text{End}(M_s)$ is $cl$-fully stable act.

(1$\rightarrow$2) By corollary (2.11).

The following proposition explain the characterization of closed stable subact:

**Proposition (2.13):** Let $M_s$ be an $S$-act and $M_s = A \cup B$, where $A$ and $B$ are two subacts of $M_s$. If $N$ is closed stable subact of $M_s$, then $N = (A \cap N) \cup (B \cap N)$.

**Proof:** Let $\pi_A : M_s \rightarrow A$ and $\pi_B : M_s \rightarrow B$ be the projection maps of $M_s$ onto $A$ and $B$ respectively. Because $N$ is stable subact of $M_s$, then $\pi_A(N) \subseteq N$ and $\pi_B(N) \subseteq N$. Thus $\pi_A(N) \subseteq A \cap N$ and $\pi_B(N) \subseteq B \cap N$. Now, $N = 1_N(N) = \pi_A(N) \cup \pi_B(N) \subseteq (A \cap N) \cup (B \cap N)$. The other direction of the inclusion is obvious. Hence $N = (A \cap N) \cup (B \cap N)$.

In the following, we introduce a class of acts larger than the class of closed fully stable acts:

**Definition (2.14):** Let $M_s$ be an $S$-act. A closed subact $N$ of $M_s$ is called closed pseudo stable if $f(N) \subseteq N$ for each $S$-monomorphism $f : N \rightarrow M_s$ and $M_s$ is called closed fully pseudo stable act (for simply $cl$-fully pseudo stable) if each subact is closed pseudo stable.

The proof is essentially the same as the corresponding result in [6], where proved that fully stable act and fully pseudo stable are coincide.

**Proposition (2.15):** Every closed fully pseudo stable reversible act is closed fully stable act.

**Definition (2.16):** An $S$-act is called terse if distinct subacts are not isomorphic.

The following proposition show that the concepts of terse and closed fully pseudo stable are coincide, the proof of the following proposition by lemma(3.11) in [4].

**Proposition (2.17):** An $S$-act is $cl$-fully pseudo stable if and only if it is terse.
3-Acts related to cl-fully stable acts:

In the following proposition we try to put some light on the relation between cl-fully stable act and quasi injective, where it gives an answer for the equation: when quasi injective acts are cl-fully stable?

Before this proposition we need the following concept. Recall that an S-act is called multiplication if each subsystem of M is of the form MI , for some right ideal I of S . This is equivalent to saying that every principal subsystem is of this form [7].

In fact, since there is no relation between multiplication acts and cl-fully stable acts, so we can use it as a condition for the following proposition:

**Proposition (3.1):** Let M be multiplication S-act over commutative monoid . If M is quasi injective, then M is cl-fully stable act.

**Proof:** Let N be any closed subact of M and f:N:M be any S-homomorphism . Since M is multiplication, so N = IM for some ideal I of S . By quasi injectivity of M , f can be extended to S-homomorphism g :M:M . Now, f(N)=g(N)=g(IM) = Ig(M) ⊆ IM=N .

**Proposition (3.2):** Let M be multiplication S-act over commutative monoid . If M is pseudo injective, then M is cl-fully pseudo stable act.

**Proof:** The proof is essentially the same as the proposition (3.1) by replacing the homomorphism f :N → M by S-monomorphism .

The following proposition explains relation between closed fully stable S-act and Baer criterion, but before we need the following concept:

**Definition (3.3):** Let N be a subact of some act M . We say that N satisfies Baer criterion, if for every S-homomorphism f:N:M , there exists an element s ∈ S such that f(n) = ns for each n ∈ N . An S-act M is said to satisfy Baer criterion if every subact of M satisfies Baer criterion.

**Proposition (3.4):** If M is closed fully stable S-act, then M satisfies Baer criterion for closed subacts (where S is a commutative monoid).

**Proof:** Let N be a closed subact of M and f :N → M an S-homomorphism. Since N is stable, we have f(N) ⊆ N and hence f(n) ∈ N , which implies that f(n) ∈ M , but N is closed (this means has no proper essential extension) , so there is t ∈ S such that f(n) = nt. Let w ∈ N, hence w = nr for some r ∈ S and hence f(w) ∈ N . So f(w) = f(nr) = f(n)r = (nt)r = n(tr) = x(n)t = (nt)t = wt. Hence there is t ∈ S such that f(w) = wt for every w N . Thus Baer criterion holds for closed subacts.
References


